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On a Generalization of Jackson's Theorem for Best Approximation

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INTRODUCTION

A generalization of the uniform distance

$$\rho(f, g) = \sup_{x \in \Omega} | f(x) - g(x)|$$

between two bounded and continuous functions defined on the set Ω of the real axis is the Hausdorff distance $r(f, g; \alpha)$ with a parameter $\alpha > 0$, which is defined as follows [1]:

Let

$$| f(x) - g(x)|_\alpha = \max\{\inf_{t \in \Omega} \max[(1/\alpha)| x - t |, | f(x) - g(t)|], \inf_{t \in \Omega} \max[(1/\alpha)| x - t |, f(t) - g)x)|]\};$$

then

$$r(f, g; \alpha) = \sup_{x \in \Omega} | f(x) - g(x)|_\alpha .$$

The distance $r(f, g; \alpha)$ can be considered as a generalization of $\rho(f, g)$ since

$$\rho(f, g) = r(f, g; 0) = \lim_{\alpha \to 0} r(f, g; \alpha).$$

It is easy to obtain the following relation between $r(f, g; \alpha)$ and $\rho(f, g)$ for every $\alpha > 0$:

$$r(f, g; \alpha) \leqslant \rho(f, g) \leqslant r(f, g; \alpha) + \omega(\alpha r(f, g; \alpha)), \tag{1}$$

where $\omega(\delta)$ is the modulus of continuity of $f(x)$ or of $g(x)$.

Let $\Omega = R_1$ be the real axis and T_n the set of trigonometrical polynomials of degree $\leqslant n$. The best approximation $E_n^T(f; \alpha)$ of the function $f(x)$ relative to the Hausdorff distance with elements of T_n is defined by

$$E_n^T(f; \alpha) = \inf_{P \in T_n} r(f, P; \alpha)$$

in analogy to the best uniform approximation

$$E_n^T(f) = \inf_{P \in T_n} \rho(f, P).$$

It follows immediately from (1) that if $\omega(f; \delta)$ is the modulus of the continuous function $f(x)$, then for every $\alpha \geqslant 0$ the inequalities (2) hold:

$$E_n^T(f; \alpha) \leqslant E_n^T(f) \leqslant E_n^T(f; \alpha) + \omega(f; \alpha E_n^T(f; \alpha)). \tag{2}$$

The following estimate [1] is known for the best approximation with trigonometrical polynomials relative to Hausdorff distance with a fixed $\alpha > 0$:

$$E_n^T(f; \alpha) = O(\ln n/n). \tag{3}$$

The same estimate holds also for approximation with algebraic polynomials and with entire functions of exponential type [2, 3].

The classic result of Jackson for the best uniform approximation is [4]:

$$E_n^T(f) = O(\omega(f; n^{-1})). \tag{4}$$

The estimate (3) cantnot be considered as a generalization of (4), because we can not obtain (4) directly from (3).

This communication aims at the perfection of the estimate (3) in such a way, that (3) will imply (4).

It turns out that an estimate of the kind

$$E_n^T(f; \alpha) = O(\ln(\alpha n \omega(f; n^{-1}))/\alpha n) \tag{5}$$

can be found (naturally, if $\alpha n \omega(f; n^{-1}) \geqslant e$).

For fixed α the estimate (5) in the set of all continuous bounded functions is of the same order $O(\ln n/n)$ with respect to n as (3), but immediately from (5) we obtain (4). Actually, since α is an arbitrary positive number, we can take $\alpha = e/n\omega(f; n^{-1})$. Then from (2) and (5) follows

$$E_n^T(f) = O(\omega(f; n^{-1})/e) + O(\omega(f; n^{-1})) = O(\omega(f; n^{-1})).$$

The following exposition is devoted to the estimates of the type (5).

Section 1 contains some auxiliary assertions.

In Section 2 we obtain an estimate of the type (5) for the best approximation $E_\nu(f; \alpha)$ relative to Hausdorff distance with parameter α of a bounded and continuous function on the real axis by means of entire functions of exponential type.

In Section 3 we treat the similar case about the approximation of 2π-periodic continuous functions by means of trigonometrical polynomials.

1. In studying the Hausdorff distance between functions the most suitable modulus turned out to be that of the nonmonotonicity [1]:

$$\mu(f; \delta) = \tfrac{1}{2} \sup_{\substack{x' \leqslant x \leqslant x'' \\ x'' - x' \leqslant \delta}} [|f(x') - f(x)| + |f(x'') - f(x)| - |f(x'') - f(x')|].$$

Immediately it is ascertained that $\mu(f; \delta) \leqslant \omega(f; \delta)$. The following assertion is valid.

LEMMA 1. *If the function $f(x)$ is monotonic in every interval with length* $\leqslant \delta_0$, then $\mu(f; \delta_0) = 0$.

The proof of the lemma follows immediately from the definition of $\mu(f; \delta)$.

Besides the definition given at the beginning, the Hausdorff distance with parameter α can be defined also in the following way.

Let

$$h(f, g; \alpha) = \sup_{x \in \Omega} \inf_{t \in \Omega} \max[(1/\alpha)| x - t |, | f(x) - g(t)|];$$

then

$$r(f, g; \alpha) = \max\{h(f, g; \alpha), h(g, f; \alpha)\}. \tag{6}$$

The two definitions are obviously equivalent. We shall call the number $h(f, g; \alpha)$ the one-way distance between $f(x)$ and $g(x)$, and $h(g, f; \alpha)$ the one-way distance between $g(x)$ and $f(x)$.

The Hausdorff distance with parameter α between two functions $f(x)$ and $g(x)$ can be expressed by the one-way distance between $f(x)$ and $g(x)$ and the modulus of nonmonotonicity of $f(x)$.

LEMMA 2. *If $f(x)$ and $g(x)$ are continuous functions, then*

$$r(f, g; \alpha) \leqslant h(g, f; \alpha) + \mu(f; 4\alpha h(g, f; \alpha)).$$

Proof. Let us denote

$$\delta = h(g, f; \alpha), \qquad \theta = \delta + \mu(f; 4\alpha h(g, f; \alpha)).$$

According to (6), it is enough to show that

$$h(f, g; \alpha) \leqslant \theta,$$

i.e., for every point $(x, f(x))$ there exists t such that

$$\max[(1/\alpha)|\, x - t\,|, |\, f(x) - g(t)|] \leqslant \theta.$$

Let us assume the contrary, that there exists x_0, such that no point of the graph of $g(x)$ lies in the rectangle

$$x \in [x_0 - \alpha(\theta + \epsilon), x_0 + \alpha(\theta + \epsilon)] = \Delta_\theta$$
$$y \in [f(x_0) - \theta - \epsilon, f(x_0) + \theta + \epsilon], \quad \epsilon > 0.$$

Then for every $x \in \Delta_\theta$ there will be fulfilled either the inequality

$$g(x) > f(x_0) + \theta + \epsilon$$

or the inequality

$$g(x) < f(x_0) - \theta - \epsilon. \tag{7}$$

Let us suppose more specifically, that for each $x \in \Delta_\theta$ the inequality (7) holds and therefore

$$g(x_0 - \alpha\delta) < f(x_0) - \theta - \epsilon,$$
$$g(x_0 + \alpha\delta) < f(x_0) - \theta - \epsilon.$$

But then by the definition of $h(g, f; \alpha)$ it follows that there exist x' and x'' for which

$$|\, x_0 - \alpha\delta - x'\,| \leqslant \alpha\delta, \qquad |\, x_0 + \alpha\delta + x''\,| \leqslant \alpha\delta$$

(i.e.,

$$x' < x_0 < x'', \qquad x'' - x' \leqslant 4\alpha\delta)$$

and

$$f(x') \leqslant f(x_0) - \theta - \epsilon + \delta = f(x_0) - \mu(f; 4\alpha\delta) - \epsilon,$$
$$f(x'') \leqslant f(x_0) - \theta - \epsilon + \delta = f(x_0) - \mu(f; 4\alpha\delta) - \epsilon.$$

Then

$$\mu(f; 4\alpha\delta) \geqslant (1/2)[|\, f(x') - f(x_0)| + |\, f(x'') - f(x_0)| - |\, f(x') - f(x'')|]$$
$$\geqslant \min[f(x_0) - f(x'), f(x_0) - f(x'')] \geqslant \mu(f; 4\alpha\delta) + \epsilon,$$

which contradicts the inequality $\epsilon > 0$. Thus the lemma is proved.

LEMMA 3. *Let $f(x)$ be a continuous bounded (2π-periodic) function on the real axis and let $\delta > 0$. Then there exists a continuous bounded (2π-periodic) function on the real axis $g(x)$, for which*

$$r(f, g; \alpha) \leqslant 4\delta/\alpha, \qquad \mu(g; \delta) = 0,$$

$$\omega(g; \theta) \leqslant 10\omega(f; \theta) \qquad \text{for} \quad \theta \leqslant 2\delta.$$

Proof. Let us denote $x_k = k\delta$; $k = 0, \pm 1, \pm 2,...,$

$$m_k = \min_{|x_k - x| \leqslant 2\delta} f(x), \qquad M_k = \max_{|x_k - x| \leqslant 2\delta} f(x).$$

Let us consider the continuous function $g(x)$, defined as follows:

$$g(x) = \begin{cases} m_{4k} & \text{for} \quad x_{4k-2} \leqslant x \leqslant x_{4k-1}, \\ M_{4k} & \text{for} \quad x_{4k} \leqslant x \leqslant x_{4k+1}, \\ \text{linear} & \text{for} \quad x_{2k-1} \leqslant x \leqslant x_{2k}. \end{cases}$$

According to Lemma 1 $\mu(g; \delta) = 0$, and by the definition of $r(f, g; \alpha)$ it follows that $r(f, g; \alpha) \leqslant 4\delta/\alpha$. On the other hand, if we denote

$$M = \max_k \max[M_{4k} - m_{4k}, M_{4k} - m_{4k+4}],$$

then obviously $\omega(f; 8\delta) \geqslant M$. Thus for $\theta \leqslant 2\delta$ we shall have

$$\omega(f; 8\delta) \leqslant (1 + 8\delta/\theta)\, \omega(f; \theta) \leqslant (10\delta/\theta)\, \omega(f; \theta)$$

or

$$\omega(f; \theta) \geqslant \theta M/10\delta. \tag{8}$$

By the definition of $g(x)$ it follows immediately that for $\theta \leqslant 2\delta$ we have

$$\omega(g; \theta) \leqslant \theta M/\delta. \tag{9}$$

From (8) and (9) follows

$$\omega(g; \theta) \leqslant 10\omega(f; \theta) \qquad \text{for} \quad \theta \leqslant 2\delta.$$

The 2π-periodic case is considered in a similar way.

2. Let us consider first the approximation of bounded continuous functions on the real axis by means of entire functions of exponential type relative to the Hausdorff distance $r(f, g; \alpha)$ with parameter $\alpha > 0$.

Let $f(x)$ be a continuous bounded function on the real axis, and let A_ν denote the class of all entire functions of the exponential type ν [5, 6]. The best approximation of $f(x)$ by elements of A_ν relative to the Hausdorff distance with parameter α is defined through

$$E_\nu(f;\alpha) = \inf_{g \in A_\nu} r(f, g; \alpha).$$

We shall find an estimate for $E_\nu(f;\alpha)$ in terms of α, ν and the modulus of continuity $\omega(f;\delta) = \sup_{|x-y|\leqslant\delta} |f(x) - f(y)|$ of the function $f(x)$.

Let us consider the entire function of exponential type ν:

$$g_\nu(x) = (\sin mx/mx)^r,$$

where

$$mr \leqslant \nu, \qquad r = 2s, \qquad s > 1, \qquad s - \text{integral}.$$

Let μ_ν be a norming factor defined by the condition

$$\mu_\nu \int_{-\infty}^{\infty} g_\nu(x)\,dx = 1 \tag{10}$$

Then

$$f_\nu(x) = \mu_\nu \int_{-\infty}^{\infty} f(t)\, g_\nu(x-t)\,dt$$

is an entire function of exponential type ν[5].

Let us find an estimate for the Hausdorff distance $r(f, f_\nu; \alpha)$ with a parameter α by means of $\omega_2(f;\delta)$,

$$\omega_2(f;\delta) = \sup_x \sup_{0<h\leqslant\delta} |f(x+h) - 2f(x) + f(x-h)|$$

and the modulus of nonmonotonicity $\mu(f;\delta)$ of $f(x)$.

At first we shall estimate the one-way distance between $f_\nu(x)$ and $f(x)$. We have

$$\begin{aligned} f_\nu(x) - f(x) &= \mu_\nu \int_{-\infty}^{\infty} [f(x-t) - f(x)]\, g_\nu(t)\,dt \\ &= \mu_\nu \int_0^{\infty} [f(x+t) - 2f(x) + f(x-t)]\, g_\nu(t)\,dt. \end{aligned}$$

If $u \in [-\delta, \delta]$, then

$$\begin{aligned} 2[\min_{|x-t|\leqslant\delta} (f(t) - f(x))] &\leqslant f(x+u) - 2f(x) + f(x+u) \\ &\leqslant 2[\max_{|x-t|\leqslant\delta} (f(t) - f(x))] \end{aligned}$$

and therefore from (10) follows

$$\min_{|x-u|\leqslant\delta}(f(u)-f(x)) \leqslant \mu_\nu \int_0^\delta [f(x+t)-2f(x)+f(x-t)]\, g_\nu(t)\, dt$$
$$\leqslant \max_{|x-u|\leqslant\delta}(f(u)-f(x)).$$

Then through the continuity of $f(x)$ we infer that there exists a point y_x, $|x-y_x|\leqslant\delta$, such that

$$f(y_x)-f(x) = \mu_\nu \int_0^\delta [f(x+t)-2f(x)+f(x-t)]\, g_\nu(t)\, dt. \tag{12}$$

From (11) and (12) we obtain

$$h(f_\nu, f; \alpha) \leqslant \sup_x \max\{\alpha^{-1}|x-y_x|, |f(y_x)-f_\nu(x)|\} \leqslant \max\{\alpha^{-1}\delta, \varphi_\nu(\delta)\}, \tag{13}$$

where

$$\varphi_\nu(\delta) = \sup_x \left| \mu_\nu \int_\delta^\infty [f(x+t)-2f(x)+f(x-t)]\, g_\nu(t)\, dt \right|.$$

Therefore we have to estimate $\varphi_\nu(\delta)$. Let us note, that

$$\omega_2(f; s\lambda) \leqslant (1+\lambda)^2\, \omega_2(f; s)$$

and consequently

$$\omega_2(f; t) \leqslant (1+kt)^2\, \omega_2(f; 1/k). \tag{14}$$

From (10) we obtain for μ_ν

$$\mu_\nu^{-1} = \int_{-\infty}^\infty (\sin mx/mx)^r\, dx \geqslant 2\int_0^{\pi/2m} (\sin mx/mx)^r\, dx \geqslant \frac{2}{m}\left(\frac{2}{\pi}\right)^{r-1}$$

Therefore,

$$\mu_\nu \leqslant (m/2)(\pi/2)^{r-1}. \tag{15}$$

From (14) and (15) we obtain

$$\begin{aligned}
|\varphi_\nu(\delta)| &\leqslant \mu_\nu \int_\delta^\infty \omega_2(f; t)\, g_\nu(t)\, dt \\
&\leqslant \mu_\nu \omega_2(f; k^{-1})\, m^{-r} \int_\delta^\infty (kt+1)^2\, t^{-r}\, dt \\
&\leqslant \frac{1}{2}\left(\frac{\pi}{2m}\right)^{r-1} \omega_2(f; k^{-1}) \left[\frac{k^2}{(r-3)\,\delta^{r-3}} + \frac{2k}{(r-2)\,\delta^{r-2}} + \frac{1}{(r-1)\,\delta^{r-1}}\right] \\
&\leqslant \frac{1}{2(r-3)}\left(\frac{\pi}{2m}\right)^{r-1} \frac{(k\delta+1)^2}{\delta^{r-1}}\, \omega_2(f; k^{-1}).
\end{aligned} \tag{16}$$

From (13) and (16) follows

$$h(f_\nu, f; \alpha) \leqslant \max\left\{\alpha^{-1}\delta, \frac{1}{2(r-3)}\left(\frac{\pi}{2m}\right)^{r-1}\frac{(k\delta+1)^2}{\delta^{r-1}}\,\omega_2(f; k^{-1})\right\}. \tag{17}$$

Now, let us set in (17)

$$\begin{aligned} \delta &= 2\pi\nu^{-1}e^{1/2}\ln(\alpha\nu\omega_2(f; \nu^{-1})), \qquad k = \nu, \\ r &= 2[\ln(\alpha\nu\omega_2(f; \nu^{-1}))] + 2, \qquad m = \nu/4\ln(\alpha\nu\omega_2(f; \nu^{-1})); \end{aligned} \tag{18}$$

where [ ] denotes the integral part. (We suppose that $\alpha\nu\omega_2(f; \nu^{-1}) \geqslant e$, which further on we shall consider everywhere fulfilled. If $\alpha\nu\omega_2(f; \nu^{-1}) < e$, then we can substitute 1 for $\ln(\alpha\nu\omega_2(f; \nu^{-1}))$ everywhere).

Then from (17) it follows that

$$h(f_\nu, f; \alpha) \leqslant c\ln(\alpha\nu\omega_2(f; \nu^{-1}))/\alpha\nu \tag{19}$$

where c is an absolute constant.

From (19) and Lemma 2 follows Theorem 1.

THEOREM 1. *If $f(x)$ is a continuous and bounded function on the real axis, then*

$$r(f, f_\nu; \alpha) \leqslant c\,\frac{\ln(\alpha\nu\omega_2(f; \nu^{-1}))}{\alpha\nu} + \mu\left(f; \frac{4c\ln(\alpha\nu\omega_2(f; \nu^{-1}))}{\nu}\right), \tag{20}$$

where c is an absolute constant and

$$f_\nu(x) = \mu_\nu\int_{-\infty}^{\infty} f(t)\left(\frac{\sin m(x-t)}{m(x-t)}\right)^r dt, \mu_\nu, m, r$$

are defined by (10) and (18), and $\alpha\nu\omega_2(f; \nu^{-1}) \geqslant e$.

Now we shall obtain the desired estimate of $E_\nu(f; \alpha)$. Let ν be fixed. By Lemma 3 there exists a continuous function $g_\delta(x)$ such, that

$$\begin{aligned} r(f, g_\delta; \alpha) &\leqslant 4\delta/\alpha, \qquad \mu(g_\delta; \delta) = 0, \\ \omega(g_\delta; \theta) &\leqslant 10\omega(f; \theta) \qquad \text{for} \quad \theta \leqslant 2\delta. \end{aligned} \tag{21}$$

From (20) and (21) we obtain for $\delta = 4c\nu^{-1}\ln(20\alpha\nu\omega(f; \nu^{-1}))$

$$\begin{aligned} E_\nu(f; \alpha) &\leqslant r(f, g_\delta; \alpha) + E_\nu(g_\delta; \alpha) \\ &\leqslant \frac{16c\ln(20\alpha\nu\omega(f; \nu^{-1}))}{\alpha\nu} + \frac{c\ln(\alpha\nu\omega_2(g_\delta; \nu^{-1})}{\alpha\nu} \\ &\quad + \mu\left(g_\delta; \frac{4c\ln(\alpha\nu\omega_2(g_\delta; \nu^{-1}))}{\nu}\right) = O\left(\frac{\ln(\alpha\nu\omega(f; \nu^{-1}))}{\alpha\nu}\right). \end{aligned} \tag{22}$$

We have used in the above inequalities

$$\mu\left(g_\delta\,;\frac{4c\ln(\alpha\nu\omega_2(g_\delta\,;\nu^{-1}))}{\nu}\right)\leqslant\mu(g_\delta\,;\delta)=0$$

since

$$\omega_2(g_\delta\,;\nu^{-1})\leqslant 2\omega(g_\delta\,;\nu^{-1})\leqslant 20\omega(f;\nu^{-1})$$

and also

$$\nu^{-1}\leqslant 8c\nu^{-1}\ln(20\alpha\nu\omega(f;\nu^{-1}))=2\delta.$$

(We can take $c\geqslant 1$ and as above the expressions under the logarithms $\geqslant e$, otherwise we substitute 1 for the logarithms.)

From (22) we obtain the wanted generalization of the estimate of Jackson's type for the best uniform approximation of a continuous bounded function on the real axis with entire functions of exponential type ν (see for instance [5]).

THEOREM 2. *Let $f(x)$ be a continuous bounded function on the real axis. Then*

$$E_\nu(f;\alpha)\leqslant\max\left\{\frac{c\ln(\alpha\nu\omega(f;\nu^{-1}))}{\alpha\nu},\frac{c}{\alpha\nu}\right\}\tag{23}$$

where c is an absolute constant.

Setting $\alpha=e/\nu\omega(f;\nu^{-1})$ in (23) and using (1) we obtain as a corollary of (23) the estimate of uniform approximations with entire functions of exponential type:

$$E_\nu(f)=O(\omega(f;\nu^{-1})).$$

3. Now let $f(x)$ be a 2π-periodic function. An estimate of the type (5) for its best approximations relative to the Hausdorff distance with parameter α by trigonometrical polynomials of the n-th degree is obtained in an entirely similar way to the corresponding estimate of Section 2. Therefore, we shall confine ourselves to some short notes.

One has to consider the trigonometrical polynomials of n-th degree

$$f_n(x)=\mu_n\int_{-\pi}^{\pi}f(x-t)\left(\frac{\sin mt}{m\sin t}\right)^r dt$$

where

$$\mu_n\int_{-\pi}^{\pi}\left(\frac{\sin mt}{m\sin t}\right)dt=1,\qquad mr\leqslant n,\qquad r=2s,\qquad s>1,$$

and m, s are integers.

Then by a suitable choice of the numbers m and r (the same as in (18), but m integer) we obtain

$$r(f, f_n\,; \alpha) \leqslant \frac{c \ln(\alpha n \omega_2(f; n^{-1}))}{\alpha n} + \mu \left(f; \frac{4c \ln(\alpha n \omega_2(f; n^{-1}))}{n}\right) \tag{24}$$

where c is an absolute constant.

From (24) and Lemma 3 for 2π-periodic functions, we obtain the following.

THEOREM 3. *Let $f(x)$ be a 2π-periodic continuous function. Then*

$$E_n{}^T(f; \alpha) \leqslant \max \left\{c \frac{\ln(\alpha n \omega(f; n^{-1}))}{\alpha n}, \frac{c}{\alpha n}\right\}, \tag{25}$$

where c is an absolute constant.

Setting $\alpha = e/n\omega(f; n^{-1})$ in (25) and using (2) we obtain as a corollary the classic theorem of Jackson:

$$E_n{}^T(f) = O(\omega(f; n^{-1})).$$

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